AN ALGEBRA DENSE IN ITS TILDE ALGEBRA

BY

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ABSTRACT

We construct a closed set E in the circle such that $A(E)$ is dense in, but not equal to, its tilde algebra.

Introduction

We follow the notation and definitions of [2, in particular Chapter IV] where the tilde algebra $\tilde{A}(E)$ is treated.

Let E be a closed subset of $\mathbb{T}$. Recall that $\tilde{A}(E)$ is the space of functions $f \in C(E)$ for which there exist sequences $\{\phi_n\} \subseteq A$ satisfying

$$\phi_n \to f \text{ uniformly on } E \tag{1.1}$$

$$\sup \| \phi_n \|_A < \infty . \tag{1.2}$$

If $f \in \tilde{A}(E)$ we write

$$\| f \|_{\tilde{A}(E)} = \inf\{\sup \| \phi_n \|_A : \{\phi_n\} \text{ satisfies } (1.1)\}.$$

It is clear that $(\tilde{A}(E), \| \ \|_{\tilde{A}(E)})$ is a Banach algebra, that $\tilde{A}(E) \supseteq A(E)$ and that $\| f \|_{\tilde{A}(E)} \leqq \|f\|_{A(E)}$ for all $f \in A(E)$. What is the relation between $A(E)$ and $\tilde{A}(E)$?

Katznelson and McGehee have shown [4] (see also [5]) that we can have $A(E) \neq \tilde{A}(E)$ and $\| f \|_{A(E)} = \| f \|_{\tilde{A}(E)}$ for all $f \in A(E)$. Later, Varopoulos showed [9] that we can have $A(E)$ not even closed in $\tilde{A}(E)$ (he gave another proof in [8, Sect. 2 Chap. 12]). A different proof of this result, which is further developed in this paper, was given in [7]. Varopoulos also showed that we can have $A(E) = \tilde{A}(E)$ yet $\sup_{0 \neq f \in A(E)}(\| f \|_{A(E)}/ \|f \|_{\tilde{A}(E)}) \geqq 1 + \varepsilon_0$, but he was unable to take ε_0 arbitrarily.

In this paper we prove:

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THEOREM 1. *There exists a closed set E such that $A(E) \neq \tilde{A}(E)$ yet $A(E)$ is dense in $(\tilde{A}(E), \|\ \|_{\tilde{A}(E)})$.*

THEOREM 2. *For every $K \geqq 1$ there exists a closed set E such that $A(E) = \tilde{A}(E)$ yet for some $0 \neq f \in A(E)$, $\|f\|_{A(E)} \geqq K \|f\|_{\tilde{A}(E)}$.*

REMARKS. (i) The proofs given in this paper can be modified without much trouble to yield an independent E using the methods described in [7, Sect. 9]

(ii) If E is the set described in Theorem 1, then

(a) the spectrum of $\tilde{A}(E)$ is E,

(b) $\tilde{A}(E)$ is self-adjoint.

(b) follows from (a) and from the fact that $\tilde{A}(F)$ is always self-adjoint on F for F closed. No other cases are known where $\tilde{A}(E) \neq A(E)$ and $\tilde{A}(E)$ has spectrum E or where $\tilde{A}(E) \neq A(E)$ and $\tilde{A}(E)$ is self-adjoint (see [4] and [5]).

2. Preliminaries

We start by making some simple observations on the structure of $\tilde{A}(E)$. Let E be a closed subset of $\mathbb{T}$. We denote by $A^*(E)$ the space of functions $f \in \tilde{A}(E)$ for which there exist sequences $\{\phi_n\} \subseteq A$ satisfying

$$\phi_n \to f \text{ uniformly on } E \tag{2.1}$$

$$\sup \|\phi_n\|_A < \infty \tag{2.2}$$

$$\hat{\phi}_n(j) \to 0 \text{ as } n \to \infty \text{ for every } j \in \mathbb{Z}. \tag{2.3}$$

If $f \in A^*(E)$ we write

$$\|f\|_{A^*(E)} = \inf\{\sup \|\phi_n\|_A : \{\phi_n\} \text{ satisfies (2.1) and (2.3)}\}.$$

It is clear that $(A^*(E) \|\ \|_{A^*(E)})$ is a Banach algebra and that $\|f\|_{\tilde{A}(E)} \leqq \|f\|_{A^*(E)}$ for all $f \in A^*(E)$.

A standard application of the Hahn-Banach theorem gives the following characterisations of $\tilde{A}(E)$ and $A^*(E)$..

LEMMA 3. *Assume $f \in C(E)$. Then*

(i) *$f \in \tilde{A}(E)$ if and only if for some constant C_1*

$$\left| \int f \, d\mu \right| \leqq C_1 \sup |\hat{\mu}(j)| \text{ for all } \mu \in M(E). \tag{2.4}$$

(ii) *$f \in A^*(E)$ if and only if for some constant C_2*

$$\left| \int f \, d\mu \right| \leqq C_2 \limsup_{|j| \to \infty} |\hat{\mu}(j)| \text{ for all } \mu \in M(E). \tag{2.5}$$

Moreover,

$$\| f \|_{\tilde{A}(E)} = \inf\{C_1 : (2.4) \text{ holds}\} \qquad [f \in \tilde{A}(E)]$$

$$\| f \|_{A^*(E)} = \inf\{C_2 : (2.5) \text{ holds}\} \qquad [f \in A^*(E)] .$$

LEMMA 4. *Let $f \in \tilde{A}(E)$; then there exist functions $\phi \in A$ and $f_1 \in A^*(E)$ such that $f = \phi \mid E + f_1$ and $\| f \|_{\tilde{A}(E)} = \| \phi \|_A + \| f_1 \|_{A^*(E)}$.*

PROOF. Let $\{\phi_n\} \subseteq A$ be such that $\| \phi_n \|_A \leqq \| f \|_{\tilde{A}(E)}$ and $\phi_n \mid E \to f$ uniformly on E. Extracting a subsequence if necessary we may suppose that for each j, $\hat{\phi}_n(j)$ converges to a_j, say, as $n \to \infty$. Clearly $\Sigma \mid a_j \mid \leqq \| f \|_{A(E)}$ so $\phi(t) = \Sigma\, a_j e^{ijt}$, $t \in \mathbb{T}$ defines a $\phi \in A$. Now as $n \to \infty$, $\| \phi - \phi_n \|_A + \| \phi \|_A - \| \phi_n \|_A \to 0$, $\phi_n - \phi \to f - \phi$ uniformly on E, and $(\phi_n - \phi)^\wedge (j) = \hat{\phi}_n(j) - \hat{\phi}(j) = \hat{\phi}_n(j) - a_j \to 0$. Thus

$$\| f \|_{\tilde{A}(E)} \geqq \| \phi \|_A + \| f - \phi \|_{A^*(E)}$$

and since the reverse inequality is obvious the lemma follows.

LEMMA 5. *Suppose $f \in \tilde{A}(E)$, $\eta > 0$ given such that for every $\varepsilon > 0$ we can find a $\mu \in M(E)$ with*

$$\limsup_{|j| \to \infty} |\hat{\mu}(j)| \leqq (1 - \eta) \sup |\hat{\mu}(j)| \tag{2.6}$$

$$\left| \int f \, d\mu \right| \geqq (1 - \varepsilon) \sup |\hat{\mu}(j)| \, \| f \|_{\tilde{A}(E)}. \tag{2.7}$$

Then f is the restriction to E of a function $\phi \in A$ such that $\| \phi \|_A = \| f \|_{\tilde{A}(E)}$.

PROOF. Write $f = \phi \mid E + f_1$ as in Lemma 4. If μ satisfies (2.6) and (2.7) we have

$$(1 - \varepsilon) \| f \|_{A(E)} = \frac{1}{\sup |\hat{\mu}(j)|} \left| \int f d\mu \right|$$

$$\leqq \frac{1}{\sup |\hat{\mu}(j)|} \left(\left| \int \phi d\mu \right| + \left| \int f_1 d\mu \right| \right) \leqq \| \phi \|_A + (1 - \eta) \| f \|_{A^*(E)}$$

using (2.6).

Since $\varepsilon > 0$ is arbitrary it follows that

$$\| \phi \|_A + \| f_1 \|_{A^*(E)} = \| f \|_{\tilde{A}(E)} \leqq \| \phi \|_A + (1 - \eta) \| f \|_{A^*(E)}$$

so $\| f_1 \|_{A^*(E)} = 0$ and the lemma follows.

COROLLARY 6. *If the assumptions of Lemma 5 are satisfied for all $f \in \tilde{A}(E)$, then $\tilde{A}(E) = A(E)$ isometrically.*

REMARK. Corollary 6 is a generalization of a result of Krein (see [2, p. 46]). It will play an important part in the proof of Theorem 2.

The lemmas that follow show how we can derive Theorem 1 from a modified version of Theorem 2.

LEMMA 7. *Let* $E = \bigcup_{n=0}^{\infty} E_n$ *where* $E_0 = \{0\}$, $E_n \subseteq [0, 2^{-n}]$ *are disjoint closed sets and let* Λ_n, *where* $n \geqq 1$, *form disjoint sets of integers. Suppose that for each* $f_n \in A^*(E_n)$ *we can find a* $\mu_n \in M(E_n)$ *such that*:

$$\sup |\hat{\mu}_n(j)| \leqq 10^4 \tag{2.8}$$

$$\hat{\mu}_n(j) \to 0 \text{ as } |j| \to \infty \text{ on } \mathbb{Z} \backslash \Lambda_n \tag{2.9}$$

$$\int f_n \, d\mu_n \geqq \| f_n \|_{A^*(E_n)}. \tag{2.10}$$

Then if $f \in A^*(E)$ *we have, writing* $f_n = f \,|\, E_n$,

$$\Sigma \, \| f_n \|_{A^*(E_n)} \leqq 10^4 \, \| f \|_{A^*(E)}.$$

PROOF. By a slight abuse of notation we take μ_n to be a measure satisfying (2.8), (2.9), (2.10). Then writing $\mu = \Sigma_1^N \mu_n$ we have $\limsup_{|j| \to \infty} |\hat{\mu}(j)| \leqq 10^4$ and so by Lemma 3

$$\sum_1^N \| f_n \|_{A^*(E)} \leqq \sum_1^N \int f_n d\mu_n = \int f d\mu \leqq 10^4 \, \| f \|_{A^*(E)}.$$

Since N was arbitrary the lemma follows.

As temporary notation we shall call a sequence $\{E_n\}$ of closed subsets of $\mathbb{T}$ well separated if for every n there exists a function $\psi_n \in A$ such that $\| \psi_n \| \leqq 3$, $\psi_n(t) = 1$ in a neighborhood of E_n and $\psi_n(t) = 0$ in a neighborhood of $\bigcup_{k \neq n} E_k \cup \{0\}$. A simple sufficient condition for the $\{E_n\}_{n=1}^{\infty}$ of Lemma 7 to be well separated is that $E_n \subseteq 2^{-n-1} + [-2^{-n-6}, 2^{-n-6}]$.

LEMMA 8. *Suppose that the conditions of Lemma 7 are satisfied and in addition that* $\{E_n\}_{n=1}^{\infty}$ *is well separated. Then*

(i) *If* $f \in A^*(E)$ *then writing* $f_n = f\psi_n$ *we obtain*

$$\| f \|_{A^*(E)} \leqq 3 \, \Sigma \, \| f_n \|_{A^*(E_n)}$$

so by Lemma 7

$$\| f \|_{A^*(E)} \sim \Sigma \, \| f_n \|_{A^*(E_n)}.$$

(ii) *If* $A(E_n) = \tilde{A}(E_n)$ *for every* n, *then* $A(E)$ *is dense in* $\tilde{A}(E)$.

(iii) If $\sup_{0\neq g\in A(E_n)}(\| g \|_{A(E_n)}/\| g \|_{\tilde{A}(E_n)})$ *is not uniformly bounded with respect to* n *then* $\tilde{A}(E) \neq A(E)$.

REMARK. Lemmas 7 and 8 establish the architecture of the set E in Theorem 1. It remains to show how to obtain the bricks E_n.

PROOF OF LEMMA 8. (i) Note first that by Lemma 7, $\| f_n \|_{A^*(E_n)} \to 0$ and so $f(0) = 0$. By the definition of $A^*(E_n)$ we can find for each N and each n a trigonometric polynomial $g_{n,N}$ such that if $1 \leqq n \leqq N$, $\| g_{n,N} \|_A \leqq \| f_n \|_{A^*(E\ )}$, $\| g_{n,N} | E - f_n \|_{C(E_n)} \leqq 2^{-N}$ and $\hat{g}_{n,N}(j) = 0$ for $|j| \leqq N$ whilst if $n \geqq N+1$ then $g_{n,N} = 0$. Trivially $\| \Sigma\, \psi_n g_{n,N} | E - f \|_{C(E)} \leqq 2^{-N} \to 0$, $(\Sigma\, \psi_n g_{n,N})^\wedge(j) \to 0$ as $N \to \infty$ for all j, $\Sigma\, \psi_n g_{n,N} \in A$ and $\| \Sigma\, \psi_n g_{n,N} \|_A \leqq \Sigma \| \psi_n \|_A \| g_{n,N} \|_A \leqq 3\, \Sigma \| f_n \|_{A^*(E_n)}$.

(ii) Let $g \in \tilde{A}(E)$ and $\varepsilon > 0$ be given. By Lemma 4 we know that $g = \phi | E + f$ where $\phi \in A$ and $f \in A^*(E)$. Set $f_n = f | E_n$. By Lemma 7, $\Sigma \| f_n \|_{A^*(E_n)}$ is convergent, hence we can find an N such that $\Sigma_N^\infty \| f_n \|_{A^*(E_n)} \leqq \frac{1}{3}\varepsilon$. Since $A(E_n) = \tilde{A}(E_n)$ we know that we can find $h_n \in A(E_n)$ with $h_n | E_n = f_n$. Set $h = \Sigma_{n=1}^N h_n \psi_n + \phi$. We have $h \in A$ and by (i),

$$\| h | E - g \|_{\tilde{A}(E)} \leqq \| h | E - g \|_{A^*(E)} \leqq 3 \sum_{n=1}^{\infty} \| h | E_n - f_n \|_{A^*(E_n)}$$

$$\leqq 3 \sum_{n=N+1}^{\infty} \| f_n \|_{A^*(E_n)} \leqq \varepsilon.$$

Thus $A(E)$ is dense in $(\tilde{A}(E), \| \ \|_{\tilde{A}(E)})$.

(iii) If $f_n \in A(E_n)$ and $f \in A$ is such that $f\psi_n = f_n$ on E_n, then $\| f\psi_n \|_{\tilde{A}(E)} \leqq 3 \| f_n \|_{\tilde{A}(E_n)}$ and $\| f\psi_n \|_{A(E)} \geqq \| f_n \|_{A(E_n)}$. Thus

$$\| f\psi_n \|_{A(E)} / \| f\psi_n \|_{\tilde{A}(E)} \geqq 1/3 \| f_n \|_{A(E_n)} / \| f_n \|_{\tilde{A}(E_n)}.$$

3. Synthesis by special measures

In this section we give some simple variations on a theme of Carl Herz. The first half of the section proves results which are more powerful than we strictly need. While these results belong to the folklore of harmonic analysis (one of us heard from J.-P. Kahane as early as 1957) we found them nowhere in print. Moreover, only when these basic ideas are well understood do the reasons behind our ad hoc adaptations of them in Lemma 12 and following become comprehensible.

DEFINITION 9. Let $\{n_k\}$ be an increasing sequence of positive integers and let $0 \leqq \varepsilon_k < 1$, $k = 1,2,\cdots$. We say that a closed subset $E \subset \mathbb{T}$ satisfies condition $H(\{n_k\}, \{\varepsilon_k\})$ if E contains every root of unity of order n_k whose distance from E is less that $(1 - \varepsilon_k)2\pi n_k^{-1}$.

Let $\Delta_{n,\varepsilon}$ be the triangular function defined by

$$\Delta_{n,\varepsilon}(t) = \max(0, 1 - n|t|/2\pi(1 - \varepsilon))$$

and let

$$v_n = \sum_{r=1}^{n} \delta_{2\pi r/n}.$$

Our discussion relies on the following three facts about $\Delta_{n,\varepsilon}$ which can be established by direct calculation (for (3.2) use Poisson's formula).

$$\hat{\Delta}_{n,\varepsilon}(j) \geqq 0, \quad \text{where } n,j \in \mathbb{Z},\ 1 > \varepsilon \geqq 0, \tag{3.1}$$

$$n \sum_{k=-\infty}^{\infty} \hat{\Delta}_{n,\varepsilon}(j + nk) = 1, \text{ where } n, j \in \mathbb{Z}, 1 > \varepsilon \geqq 0, \tag{3.2}$$

$$n\hat{\Delta}_{n,\varepsilon_n}(j) \to 1 - \lim_{m\to\infty} \varepsilon_m \text{ for all } j \in \mathbb{Z} \text{ whenever } \lim_{m\to\infty} \varepsilon_m \text{ exists.} \tag{3.3}$$

Further, since

$$\hat{v}(j) = n \text{ when } j \text{ is a multiple of } n \text{ and}$$

$$\hat{v}(j) = 0 \text{ otherwise,}$$

we see that if $S \in PM(T)$ then

$$[(S * \Delta_{n,\varepsilon})v_n]\hat{\ }(j) = n \sum_{k=-\infty}^{\infty} \hat{S}(j + nk)\, \hat{\Delta}_{n,\varepsilon}(j + nk). \tag{3.4}$$

Applying (3.1), (3.2), (3.3) and (3.4) we obtain at once the result which forms the basis of the family of theorems generically called Herz's theorem ([1]):

$$\|(S * \Delta_{n,0})v_n\|_{PM} \leqq \|S\|_{PM} \tag{3.5}$$

$$(S * \Delta_{n,0})v_n \to S \text{ weakly.} \tag{3.6}$$

Thus if a closed set $E \subseteq \mathbb{T}$ satisfies the condition $H(\{n_k\}, \{0\})$ for some infinite sequence $\{n_k\}$ we know that every $S \in PM(\mathrm{E})$ can be synthesized isometrically by discrete measures $(S * \Delta_{n_k,0})v_{n_k}$ which (as the reader will check) lie on E. In other words E is a set of isometric synthesis. (Recall that a closed set E is of isometric synthesis if the unit ball B, of $PM(E)$ is the weak closure of the unit ball $B_2 = B \cap M(E)$; it is of bounded synthesis with constant not more than η if B lies within the weak closure of ηB_2.) We now prove a slightly stronger result.

LEMMA 10. *Let E be a closed subset of $\mathbb{T}$ such that for some infinite sequence $\{n_k\}$ of integers and some sequence $\{\varepsilon_k\}$ with $0 \leqq \varepsilon_k < 1$, E satisfies $H(\{n_k\}, \{\varepsilon_k\})$.*

(i) *If* $\liminf \varepsilon_k = 0$ *then E is a set of isometric synthesis.*

(ii) *If* $\liminf \varepsilon_k < \frac{1}{2}$ *then E is a set of bounded spectral synthesis.*

REMARKS. (i) Note that if $\liminf \varepsilon_k < 1$ then the arguments used below to prove (ii) can readily be adapted to show that every pseudo-function on E is synthesizable on E.

(ii) If $n_k \to \infty$ fairly rapidly and $\varepsilon_k < \frac{1}{2}$, $\varepsilon_k \to \frac{1}{2}$ fairly slowly, there exist perfect symmetric sets E which satisfy $H(\{n_k\}, \{\varepsilon_k\})$ trivially, being at distance greater than $(1 - \varepsilon_k)2\pi n_k^{-1}$ from all the roots of unity of order n_k. Since every perfect symmetric set contains a subset of non-synthesis (see [2, p. 72]) it follows that the value $\frac{1}{2}$ in Lemma 10 is critical.

PROOF OF LEMMA 10. Condition $H(\{n_k\}, \{\varepsilon_k\})$ implies that if $S \in PM(E)$ then $(S * \Delta_{n_k, \varepsilon_k})v_k \in M(E)$. Thus (i) follows at once (just as in the case $\varepsilon_k = 0$ above) from (3.1), (3.2), (3.3) and (3.4) which give

$$(3.5)' \qquad \| (S * \Delta_{n,\varepsilon_k})v_n \|_{PM} \leqq \| S \|_{PM}$$

$$(3.6)' \qquad S * \Delta_{n_k \xi_k} v_{n_k} \to S \text{ weakly whenever } \varepsilon_k \to 0.$$

The proof of (ii) is slightly more delicate. Let us assume without loss of generality that $\varepsilon_k \to \varepsilon < 1/2$. As before $\|(S*\Delta_{n_k,\varepsilon_k})v_{n_k}\|_{PM} \leqq \| S \|_{PM}$ so $\mu_k = (S * \Delta_{n_k \varepsilon_k})v_{n_k}$ has a (not necessarily unique) weak limit point T_1 say. By (3.2), (3.3) and (3.4) we know that $T_1 = (1 - \varepsilon)S + S_1$ where $\| S_1 \|_{PM} \leqq \varepsilon \| S \|_{PM}$. Thus $S = (1 - \varepsilon)^{-1}T_1 + (1 - \varepsilon)^{-1}S_1$ where T_1 is synthesizable on E by measures with pseudo-measure norm less than $\| S \|$. Continuing inductively we obtain $S_r \in PM(E)$ in the form $S_r = (1-\varepsilon)^{-1}T_{r+1} + (1-\varepsilon)^{-1}S_{r+1}$ where $(1-\varepsilon)^{-1}T_r$ is synthesizable on E by measures with pseudo-measure norm less than $\| S_r \|_{PM}$ and $\| S_{r+1} \|_{PM} \leqq \varepsilon \| S_r \|_{PM}$. Thus $S = \sum_{r=1}^{\infty} T_r(1 - \varepsilon)^{-r}$ where T_r is synthesizable on E by measures with pseudo-measure norm less than ε^r; in particular $\| T_r(1 - \varepsilon)^{-r} \|_{PM} \leqq (\varepsilon/(1-\varepsilon))^r$. It follows that S is synthesizable by measures on E with pseudo-measure norm less than $\| S \| \sum_{r=1}^{\infty} \varepsilon^{r-1}/(1-\varepsilon)^r = \| S \|/(1-2\varepsilon)$. Thus E is of bounded synthesis with constant not more than $(1-2\varepsilon)^{-1}$.

There is one further simple remark we shall need.

LEMMA 11. *If* $\mu \in M(\mathbb{T})$ *then*

$$(3.7) \qquad \| (\mu * \Delta_{n,\varepsilon})v_n \|_{M(\mathbb{T})} \leqq \| \mu \|_{M(\mathbb{T})}$$

and, if $\varepsilon_n \to 0$, *then for every* $f \in C(\mathbb{T})$

$$\int f \, d((\mu * \Delta_{n,\varepsilon})\nu_n) \to \int f d\mu. \tag{3.8}$$

PROOF. By linearity it suffices to consider the case $\mu \geqq 0$. Under these circumstances

$$\| (\mu * \Delta_{n,\varepsilon})\nu_n \|_{M(\mathbb{T})} = \Sigma \, (\mu * \Delta_{n,\varepsilon}) \left(\frac{2\pi k}{n}\right)$$
$$= \int \Sigma \Delta_{n \cdot \varepsilon} \left(t - \frac{2\pi k}{n}\right) d\mu(t) \leqq \int d\mu = \| \mu \|_{M(\mathbb{T})}.$$

Formula (3.8) now follows from (3.7), (3.6) and the fact that $A(\mathbb{T})$ is uniformly dense in $C(\mathbb{T})$. (But (3.8) can be obtained much more simply by using thought.)

One application of Herz's theorem (see for example [3, Chap. IX, Th. VII]) is to show that, given a closed set $E \subseteq \mathbb{T}$, one can adjoin a countable set to E in such a way as to obtain a closed set of isometric synthesis. Using the same basic ideas we shall show that we can adjoin to every E a fairly thin set in such a way as to obtain a closed set satisfying the conditions of Lemma 5 together with more technical conditions connected with Lemma 7. We shall need the following trivial but useful lemma.

LEMMA 12. *Let* $\{q_k\}_{k=1}^{\infty}$ *be a sequence of positive integers such that* $q_{k+1} \gg kq_k$ *and such that* q_k *divides* q_{k+1} *for every* k *where* $k \geqq 1$. *Let* $0 < \alpha < 1$ *and write* $b_k = [\alpha q_k q_{k-1}^{-1}]$, $\tau_k = (1 + b_k)^{-1} \sum_{j=0}^{b_k} \delta_{2\pi j/q_k}$, $\sigma_n = \prod_{k=n+1}^{\infty} * \tau_k$, $G_n = \operatorname{supp} \sigma_n$. *Then for every* $n \geqq 1$,

(i) $\sigma_n \in M^+([0, \alpha q^{-1}])$ *and* $\| \sigma_n \|_M = 1$,

(ii) *there exists an* $\eta(\alpha) > 0$ *(independent of* n *and* $\{q_k\}$*) such that*

$$\limsup_{|j| \to \infty} |\hat{\sigma}_n(j)| \leqq 1 - \eta,$$

(iii) *writing* $\Lambda = \Lambda(\{q_k\}) = \bigcup_{l=1}^{\infty} (q_{l+1} - lq_l, lq_{l+1})$ *we have* $\hat{\sigma}_n(j) \to 0$ *as* $|j| \to \infty$, $|j| \notin \Lambda$.

(iv) *for all* $k, n \geqq 1$ *we have*

$$\| e^{iq_k t} - 1 \|_{C(G_n)} \leqq 3\pi\alpha.$$

REMARK. Part (iv) is interesting only for small values of α. We shall use the lemma with $\alpha = 10^{-3}$ so that $3\pi\alpha < 10^{-2}$.

PROOF OF LEMMA 12. (i) is clear. To prove (ii) and (iii) we use the estimate

$|\hat{\sigma}_n(j)| \leqq |\hat{\tau}_k(j)|$ for $q_{k-1} - (k-1)q_{k-2} \leqq |j| \leqq q_k - kq_{k-1}$ where $k \geqq n+2$, together with the remark that $\hat{\tau}_k(j) = \hat{\tau}_k(q_k - j) \sim iq_k(1 - e^{i\alpha j/q_k})/\alpha j$, where $0 < |j| \leqq \frac{1}{2}(q_k + 1)$. Since supp σ_n lies in the closure of $\bigcup_{m=n}^{\infty} \sum_{k=n}^{m}$ supp τ_k, (iv) is also clear.

Let $\{q_k\} \subseteq \mathbb{Z}^+$ satisfy the conditions of Lemma 12. Let us fix $\alpha = 10^{-3}$, $\eta = \eta(10^{-3})$ once and for all. To every closed subset E of $\mathbb{T}$ we shall associate a set $D(E, \{q_k\})$ defined as follows.

Let F_1 be the set of roots of unity of order q_1 whose distance from E is less than $2\pi/q_1$. Put $E_1 = E \bigcup (F_1 + G_2)$, where G_2 is given as in Lemma 12 with $\alpha = 10^{-3}$. We continue by induction. If E_n is defined, take F_{n+1} to be the set of roots of unity of order q_{n+1} whose distance from E_n is less than $2\pi/q_{n+1}$ and put $E_{n+1} = E_n \cup (F_{n+1} + G_{n+2})$. We define $D(E, \{q_n\})$ as the closure of $\bigcup_{n=1}^{\infty} E_n$.

LEMMA 13. *Assume that $\{q_k\} \subseteq \mathbb{Z}^+$ satisfies the conditions of Lemma 12. Write $\varepsilon_k = q_k/q_{k+1}$. Then for every closed $E \subseteq \mathbb{T}$:*

(i) *$D(E, \{q_k\})$ satisfies the condition $H(\{q_k\}, \{\varepsilon_k\})$.*

(ii) *If $\mu \in M(D(E, \{q_k\}))$, write*

$$\mu_n' = [(\mu * \Delta_{q_n, \varepsilon_n})v_{q_n}] * \sigma_{n+1}. \tag{3.9}$$

Then $\mu_n' \in M(D(E, \{q_k\}))$, $\|\mu_n'\|_{PM} \leqq \|\mu\|_{PM}$ and

$$\limsup_{|j| \to \infty} |\hat{\mu}_n(j)| \leqq (1 - \eta)\|\mu\|_{PM} \tag{3.10}$$

$$\hat{\mu}_n'(j) \to 0 \quad as \ |j| \to \infty, \ |j| \notin \Lambda(\{q_k\}) \tag{3.11}$$

and $\mu_n' \to \mu$ in the weak $\sigma(C(\mathbb{T}), M(\mathbb{T}))$ topology.

(iii). *For every $t \in D(E, \{q_k\}) \backslash E$ and every $n \geqq 1$ we have*

$$|e^{iq_n t} - 1| < 10^{-2}. \tag{3.12}$$

PROOF. (i) follows by direct inspection. (ii) is an immediate consequence of formulas (3.5), (3.7), (3.8) and Lemma 12. (iii) is a direct consequence of Lemma 12 (iv).

COROLLARY 13. *$\tilde{A}(D(E, \{q_k\})) = A(D(E, \{q_k\}))$ isometrically.*

PROOF. By Lemma 13 (ii), the hypotheses of Corollary 6 are satisfied.

4. Proof of the main theorems

Our starting point is the existence of a weak Kronecker set, in particular a weak Dirichlet set, of multiplicity. Going back to the originalproof of the result

[7] or to the very neat proof of Kaufman [6] (or a combination of the two, which seems best) we see that we have the following additional information.

LEMMA 15. *Given an interval $I \subseteq \mathbb{T}$ and a sequence $\{l_k\} \subseteq \mathbb{Z}^+$, we can find a closed set $E_0 \subseteq I$, a subsequence $\{m_j\} \subseteq \{l_j\}$ and an increasing sequence $\{p_j\} \subseteq \mathbb{Z}^+$ such that, writing $q_j = 2^{m_j}$, we have*

$$\left\| (p_{r+1} - p_r - 1) \sum_{j=p_r+1}^{p_{r+1}} e^{iq_jt} - 1 \right\|_{C(E_0)} \to 0 \text{ as } r \to \infty \tag{4.1}$$

yet there exists an $S \in PM(E_0)$ such that

$$\hat{S}(0) = \| S \|_{PM} = 1 \text{ and } \hat{S}(j) \to 0 \text{ as } |j| \to \infty. \tag{4.2}$$

We shall use Lemma 15 with sequences $\{l_j\}$ satisfying $l_{j+1} - l_j \gg j$ so that the sequence $\{q_j\}$ will automatically satisfy the assumptions of Lemma 12.

PROOF THEOREM 2. Let $K > 1$, I an interval, and $\{l_j\}$ with $l_{j+1} - l_j \gg j$ be given. Take E_0, $\{q_j\}$, and S as in the statement of Lemma 15. Since $|(S - S*\delta_t)\hat{\ }(r)| \leqq \| S \| \, |1 - \hat{\delta}_t(r)| \leqq |rt| \, \| S \| = |rt|$ and also $|(S - S*\delta_t)\hat{\ }(r)| \leqq |\hat{S}(r)| + |(S*\delta_t)\hat{\ }(r)| = 2|\hat{S}(r)|$ it follows that for all $m \geqq 1$, $\| S - S*\delta_t \| \leqq \max(m|t|, 2\sup_{|r|>m}|\hat{S}(r)|)$. But $\hat{S}(r) \to 0$ as $|r| \to \infty$ so for $|t|$ small enough, say $|t| < \beta$, we have $\| S - S*\delta_t \| \leqq 10^{-1}K^{-1}$. Take $t_0 = \pi \sum_{j>j_0} q_j^{-1}$, where j_0 is large enough to ensure $t_0 < \beta$.

We claim that $E = D(E_0, \{q_k\}) \cup D(E_0, \{q_k\}) + t_0$ satisfies the conditions of Theorem 2. Note first that by Lemma 13 (iii) and formula (4.1) it follows that

$$\limsup_{r \to \infty} \left\| (p_{r+1} - p_r - 1)^{-1} \sum_{j=p_r+1}^{p_{r+1}} e^{iq_jt} - 1 \right\|_{C(D(E_0[q_k]))} \leqq 10^{-2} \tag{4.3}$$

whilst, noticing that $e^{iq_jt_0} \to -1$ as $j \to \infty$,

$$\limsup_{r \to \infty} \left\| (p_{r+1} - p_r - 1)^{-1} \sum_{j=p_r+1}^{p_{r+1}} e^{iq_jt} + 1 \right\|_{C(D(E_0,[q_k])+t_0)} \leqq 10^{-2}. \tag{4.4}$$

In particular $D(E_0, \{q_k\}) \cap D(E_0, \{q_k\}) + t_0 = \emptyset$ so that, by Corollary 14 $A(E) = \tilde{A}(E)$ and, by Lemma 13 (i) and Lemma 10 (i), E is of synthesis.

It also follows that the function f given by $f(t) = 1$ for $t \in D(E_0, q_k)$ and $f(t) = -1$ for $t \in D(E_0, \{q_k\}) + t_0$ is well defined and belongs to $A(E)$. Since E is of synthesis, $S - S*\delta_{t_0}$ is in the dual of $A(E)$, and so

$$\begin{aligned} \| f \|_{A(E)} &\geqq \| S - S*\delta_{t_0} \|_{PM}^{-1} \langle f, S - S*\delta_{t_0} \rangle \\ &\leqq 10^2 K \langle f, S - S*\delta_{t_0} \rangle = 2 \cdot 10^2 K. \end{aligned}$$

Thus if we can show $\| f \|_{\tilde{A}(E)} \leqq 10^2$ we shall have $\| f \|_{A(E)} \geqq K \| f \|_{\tilde{A}(E)}$ and the theorem will be proved. To this end, choose $\phi \in A(\mathbb{T})$ such that $\| \phi \|_A \leqq 100$, $|\phi(t)| \leqq 1$ for all $t \in \mathbb{T}$, $\phi(t) = 1$ for $|t| \leqq 5^{-1}$, $\phi(t) = -1$ for $|t - \pi| \leqq 5^{-1}$. Clearly $\| (p_{n+1} - p_n - 1)^{-1} \sum_{j=p_n+1}^{p_{n+1}} \phi(q_j t) \| \leqq 100$; we claim that

$$\| (p_{n+1} - p_n - 1)^{-1} \sum_{j=p_n+1}^{p_{n+1}} \phi(q_j t) - f \|_{C(E)} \to 0.$$

To see this, observe that since $\| (p_{n+1} - p_n - 1) \sum_{j=p_n+1}^{p_{n+1}} e^{iq_j t} - 1 \|_{C(E_0)} \to 0$ we can find $\alpha_n \to 0$ such that for each $x \in E_0$

(4.5) $\quad \operatorname{card}\{p_{n+1} \geqq j \geqq p_n + 1 : |e^{iq_j x} - 1| \leqq 10^{-2}\} \leqq \alpha_n(p_{n+1} - p_n - 1).$

Now suppose (using the notation of the discussion preceding Lemma 13) that $y \in E_{n+1} \setminus E_n$. Then $y \in F_{n+1} + B_{n+2}$, thus

(i) $|e^{iq_{n+1} y} - 1| \leqq 10^{-2}$ trivially

(ii) $|e^{iq_j y} - 1| \leqq 10^{-2}$ for all $j \geqq n + 2$ (using Lemma 12(iv)), and there exists an $x \in E_n$ such that

(iii) $|e^{iq_j y} - e^{iq_j x}| = q_j |x - y| \leqq 10^{j-n-4}$.

It follows by induction that for every $y \in E_n \setminus E_0$ there exists $n \geqq k \geqq 0$ and a $w \in E_0$ such that

(iv) $|e^{iq_j y} - 1| \leqq 10^{-1}$, for all $j \geqq k + 1$, and

(v) $|e^{iq_j y} - e^{iq_j w}| \leqq 10^{-2}$ for all $k \geqq j \geqq 0$.

Thus, by (4.5),

(4.6) $\quad \operatorname{card}\{p_{n+1} \geqq j \geqq p_n + 1 : |e^{iq_j y} - 1| \geqq 10^{-1}\} \leqq \alpha_n(p_{n+1} - p_n - 1)$

for every $y \in \sum_{n=0}^{\infty} E_n$ and so for every $y \in D(E, \{q_k\})$.

Thus

$$\| (p_{n+1} - p_n - 1)^{-1} \sum_{j=p_n+1}^{p_{n+1}} \phi(q_j t) - f \|_{C(E)} \leqq 2\alpha_n \to 0$$

as $n \to \infty$ and thus $\| f \|_{\tilde{A}(E)} \leqq 100$.

Careful examination of what we have just done shows that we have in fact Lemma 16.

LEMMA 16. *Given an interval $J \subseteq \mathbb{T}$, a real number $K \geqq 1$, and a sequence $\{l_j\}$ with $l_{j+1} - l_j \gg j$, we can find a set $E = E_1 \cup E_2$ where E_1, E_2 are disjoint closed sets such that:*

(i) $A(E) = \tilde{A}(E)$.

(ii) *if* $f\,|\,E_1 = 1, f\,|\,E_2 = -1$ *then* $\| f \|_{\tilde{A}(E)} \leqq 100$ *and* $\| f \|_{A(E)}/\| f \|_{\tilde{A}(E)} \geqq K$.

(iii) *let* $\Lambda = \bigcup_{j=1}^{\infty} \{k : 2^{l_j+1} - j2^{l_j} \leqq |k| \leqq j2^{l_j+1}$; *then for every* $\mu_i \in M(E_i)$ *we can find a* $\mu_{in} \in M(E_i)$ *such that*

$$\sup |\hat{\mu}_{in}(j)\| \leqq \| \mu_i \|_{PM} \tag{4.7}$$

$$\hat{\mu}_{in}(j) \to 0 \ \ as \ \ |j| \to 0 \ \ on \ \ \mathbb{Z} \setminus \Lambda \tag{4.8}$$

$$\mu_{in} \to \mu_i \ \ weakly \ \ as \ \ n \to \infty, \ for \ \ i = 1,2. \tag{4.9}$$

PROOF. Use Lemma 13(ii).

LEMMA 17. *Under the conditions of Lemma* 16 *we know that for every* $\mu \in M(E)$ *we can find a* $\mu_n \in M(E)$ *such that*

$$\sup |\hat{\mu}_n(j)| \leqq 10^3 \| \mu \|_{PM} \tag{4.7)'$$

$$\hat{\mu}_n(j) \to 0 \ \ as \ \ |j| \to 0 \ \ on \ \ \mathbb{Z} \setminus \Lambda \tag{4.8)'$$

$$\mu_n \to \mu \ \ weakly \ \ as \ \ n \to \infty. \qquad \text{for } i = 1, 2. \tag{4.9)'$$

PROOF. Let $\mu_i = \mu\,|\,E_i$, for $i = 1,2$, and set $\mu_n = \mu_{1n} + \mu_{2n}$. Formulas (4.8)′ and (4.9)′ follow directly from (4.8) and (4.9). On the other hand we have by Lemma 3(i) that

$$\begin{aligned} |\hat{\mu}_1(r) - \hat{\mu}_2(r)| &= |\langle \mu, fe^{irt} \rangle| \leqq \| \mu \|_{PM} \| fe^{irt} \|_{\tilde{A}(E)} \\ &= \| \mu \|_{PM} \| f \|_{\tilde{A}(E)} \leqq 100 \| \mu \|_{PM} \end{aligned}$$

and trivially, $|\hat{\mu}_1(r) + \hat{\mu}_2(r)| = |\hat{\mu}\,(r)| \leqq \| \mu \|_{PM} \leqq 0$ so that $\| \mu_1 \|_{PM}, \| \mu_2 \|_{PM} \leqq 100 \| \mu \|_{PM}$ and (4.7)′ follows from (4.7).

LEMMA 18. *Suppose the conditions of Lemma* 17 *hold. Then for each* $f \in A(E)$ *we can find* $\mu' \in M(E_n)$ *such that*

$$\sup |\hat{\mu}'(j)| \leqq 10^4 \tag{4.7)''$$

$$\hat{\mu}'(j) \to 0 \ \ as \ \ |j| \to \infty \ \ on \ \ \mathbb{Z} \setminus \Lambda \tag{4.8)''$$

$$\int f d\mu' \geqq \| f \|_{A^*(E)}. \tag{4.9)''$$

PROOF. Pick a $\mu \in M(E)$ with $\| \mu \|_{PM} \leqq 10$ and $\int f d\mu \geqq 2 \| f \|_{A^*(E)}$ (this can be done by Lemma 3(ii)). Then for large enough n we can take $\mu' = (\overline{\arg \int f d\mu_n})\mu_n$.

PROOF OF THEOREM 1. Use the sets E, Λ of Lemma 16, 17, and 18 to construct suitable E_n for the application of Lemma 8(ii) and (iii).

REFERENCES

1. C. S. Herz, *Spectral synthesis for the Cantor set*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 42–43.

2. J. -P. Kahane, *Séries de Fourier absolument convergentes*, Springer Verlag, Berlin, 1970.

3. J. -P. Kahane et R. Salem, *Ensembles parfaits et séries trigonometriques*, Paris, Hermann, 1963.

4. Y. Katznelson and O. C. McGehee, *Measures and pseudomeasures on compact subsets of the line*, Math. Scand. **23** (1968), 57–68.

5. Y. Katznelson and O. C. McGehee, *Some Banach algebras associated with quotients of* $L^1(R)$, Indiana Univ. Math. J. **21** (1971), 419–436.

6. R. Kaufman, (to appear in Astérisque 5, S. M. F. 1973).

7. T. W. Körner, *A pseudo function on a Helson set*. (to appear in Astérisque 5, S. M. F. 1973).

8. L. -Å. Lindahl and F. Poulsen (Eds.), *Thin sets in Harmonic Analysis*, Decker, New York, 1971.

9. N. T. Varopoulos, *Groups of continuous functions in harmonic analysis*, Acta Math. **125** (1970), 109–154

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